B.Sc. in Computer Science and Engineering Thesis

Study to Formulate New Condition for a Graph to be Hamiltonian

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December 2014

CERTIFICATION

This thesis paper titled **"Study to Formulate New Condition for a Graph to be Hamiltonian"**, submitted by the group as mentioned below has been accepted as satisfactory in partial fulfillment of the requirements for the degree B.Sc. in Computer Science and Engineering in December 2014.

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CANDIDATES' DECLARATION

This is to certify that the work presented in this thesis paper, titled, "Study to Formulate New Condition for a Graph to be Hamiltonian", is the outcome of the investigation and research carried out by the following students under the supervision of Mohammad Kaykobad, Professor, Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology, Dhaka-1000, Bangladesh,.

It is also declared that neither this thesis paper nor any part thereof has been submitted anywhere else for the award of any degree, diploma or other qualifications.

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ACKNOWLEDGEMENT

We are thankful to Almighty Allah for his blessings for the successful completion of our thesis. Our heartiest gratitude, profound indebtedness and deep respect go to our supervisor, Mohammad Kaykobad, Professor, Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology, Dhaka-1000, Bangladesh,, for his constant supervision, affectionate guidance and great encouragement and motivation. His keen interest on the topic and valuable advices throughout the study was of great help in completing thesis.

We are especially grateful to the Department of Computer Science and Engineering (CSE) of Military Institute of Science and Technology (MIST) for providing their all out support during the thesis work.

Finally, we would like to thank our families and our course mates for their appreciable assistance, patience and suggestions during the course of our thesis.

Dhaka
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ABSTRACT

The subject of this thesis is the Hamiltonian Cycle problem, which is of interest in many areas including graph theory, algorithm design, and computational complexity. Named after the famous Irish mathematician Sir William Rowan Hamilton, a Hamiltonian Cycle within a graph is a simple cycle that passes through each vertex exactly once. The goal of our thesis was to find formula for a graph to be hamiltonian. There is already a formula generated for a two cross over graph to be hamiltonian. We are trying to find a condition for three cross over graph to be hamiltonian. First of this thesis provides scope and some literature reviews. The second gives preliminaries and sufficent condition for hamiltonicity. And the third gives some new condition of hamiltonian graph.

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LIST OF ABBREVIATION

GTPL : Graph Theoretic Language
GASP : Graph Algorithm Software Package
GEA : Graphic Extended ALGOL
GIRL : Graph Information Retrieval Language
GC : Graph Coloring
PE : Precoloring Extension
LC : List Coloring
MSC : Minimum Sum Coloring
GCA : Graph Construction Algorithm
HC : Heuristic Search
GS : Geometric Spanner
IC : Interference Graph
AT : Author Topic
CH : Connected Hypergraphs
DI : Deviation Inequality
TN : Transversal Number
TFG : Triangle-free Graph
IRIV : Independent Random Indicator Variables
DG : Directed Graph
VS : Vertex Set
EV : End Vertices
ES : Edge Set
SG : Simple Graph
VL : Vertex Labeled
WG : Wheel Graph
HG : Hamiltonian Graph
ID : Internally Disjoint
IV : Internal Vertex
CG : Complete Graph
SCC : Strongly Connected Component
CN : Closed Neighborhood
SGT : Spectral Graph Theory
VC : Vertex Connectivity

EC : Edge Connectivity

BC : Biconnected Component

CHAPTER 1
INTRODUCTION

1.1 Overview

The subject of this thesis is the Hamiltonian Cycle problem. Formally, a Hamiltonian Cycle is a cycle which passes through every vertex in a graph exactly once. The Hamiltonian Cycle problem is that of determining whether a given graph contains a Hamiltonian Cycle, and such graphs are termed to be Hamiltonian. In the mathematical field of graph theory, a Hamiltonian path (or traceable path) is a path in an undirected or directed graph that visits each vertex exactly once. A Hamiltonian cycle (or Hamiltonian circuit) is a Hamiltonian path that is a cycle.

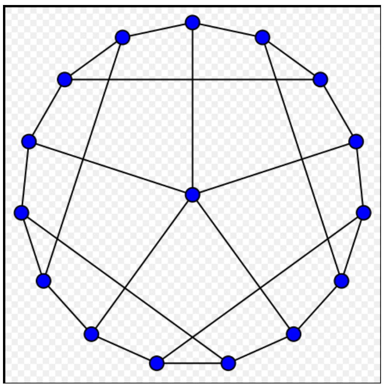

Figure 1.1: A hamiltonian graph constructed by Lindgren (1967)

Hamiltonian paths and cycles and cycle paths are named after **William Rowan Hamilton** who invented the icosian game, now also known as Hamilton's puzzle, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron. Hamilton solved this problem using the icosian calculus, an algebraic structure based on roots of unity with many similarities to the quaternions (also invented by Hamilton). This solution does not generalize to arbitrary graphs. However, despite being named after Hamilton, Hamiltonian cycles in polyhedra had also been studied a year earlier by Thomas Kirkman.

BondyChvtal theorem states that a graph is Hamiltonian if and only if its closure is Hamiltonian. As complete graphs are Hamiltonian, all graphs whose closure is complete are Hamiltonian, which is the content of the following earlier theorems by Dirac and Ore.

Dirac (1952) A simple graph with n vertices $(n \geq 3)$ is Hamiltonian if every vertex has degree n / 2 or greater.

Ore (1960) A graph with n vertices $(n \geq 3)$ is Hamiltonian if, for every pair of non-adjacent vertices, the sum of their degrees is n or greater (see Ore's theorem). The following theorems can be regarded as directed versions:

Ghouila-Houiri (1960) A strongly connected simple directed graph with n vertices is Hamiltonian if every vertex has a full degree greater than or equal to n.

Meyniel (1973) A strongly connected simple directed graph with n vertices is Hamiltonian if the sum of full degrees of every pair of distinct non-adjacent vertices is greater than or equal to $2n - 1$. The number of vertices must be doubled because each undirected edge corresponds to two directed arcs and thus the degree of a vertex in the directed graph is twice the degree in the undirected graph.

Examples:

1.A complete graph with more than two vertices is Hamiltonian

2.Every cycle graph is Hamiltonian

3.Every tournament has an odd number of Hamiltonian paths (Rdei 1934)

4.Every platonic solid, considered as a graph, is Hamiltonian

Both the Hamiltonian path and cycle problems are NP-complete. So in some sense they are mathematically equivalent. But in application, having a Hamiltonian cycle is more restrictive than having a Hamiltonian path. Obviously every graph with a Hamiltonian cycle has a hamiltonian path, and the converse is not true. Figure shows three examples of famous graphs that have Hamiltonian paths but no Hamiltonian cycles.

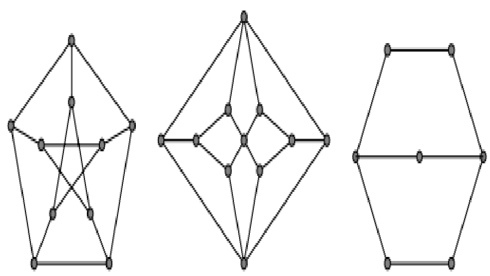

Figure 1.2: Famous graphs with Hamiltonian paths and no Hamiltonian cycles (Peterson,Herschel,Theta-7)

While the Hamiltonian cycle problem may be more famous, the author believes the Hamiltonian path problem is at least as natural and possibly more important.

1.2 Properties

Any Hamiltonian [1] cycle can be converted to a Hamiltonian path by removing one of its edges, but a Hamiltonian path can be extended to Hamiltonian cycle only if its endpoints are adjacent. There is a simple relation between the problems of finding a Hamiltonian path and a Hamiltonian cycle. In one direction, the Hamiltonian path problem for graph G is equivalent to the Hamiltonian cycle problem in a graph H obtained from G by adding a new vertex and connecting it to all vertices of G. Thus, finding a Hamiltonian path cannot be significantly slower (in the worst case, as a function of the number of vertices) than finding a Hamiltonian cycle. In the other direction, a graph G has a Hamiltonian cycle using edge uv if and only if the graph H obtained from G by replacing the edge by a pair of vertices of degree 1, one connected to u and one connected to v, has a Hamiltonian path. Therefore, by trying this replacement for all edges incident to some chosen vertex of G, the Hamiltonian cycle problem can be solved by at most n Hamiltonian path computations, where n is the number of vertices in the graph.

All Hamiltonian graphs are biconnected, but a biconnected graph need not be Hamiltonian (see, for example, the Petersen graph). [1] An Eulerian graph G (a connected graph in which every vertex has even degree) necessarily has an Euler tour, a closed walk passing through each edge of G exactly once. This tour corresponds to a Hamiltonian cycle in the line graph L(G), so the line graph of every Eulerian graph is Hamiltonian. Line graphs may have other Hamiltonian cycles that do not correspond to Euler tours, and in particular the line graph L(G) of every Hamiltonian graph G is itself Hamiltonian, regardless of whether the graph G is Eulerian.A tournament (with more than two vertices) is Hamiltonian if and only if it is strongly connected.

For the purpose of this thesis, all graphs are undirected and unweighted, with no self loops nor any multiple edges. Though similar notions may be defined for directed graphs, where each edge (arc) of a path or cycle can only be traced in a single direction (i.e., the vertices are connected with arrows and the edges traced tail-to-head). Consistent with the definition, we use the notation G = (V, E). In addition, we use n to denote no of vertices V , and m to denote no of edges E. For the readers convenience, the end of this thesis contains an alphabetized glossary.

1.3 Complexity

In the mathematical field of graph theory the Hamiltonian path problem and the Hamiltonian cycle problem are problems of determining whether a Hamiltonian path (a path in an undirected or directed graph that visits each vertex exactly once) or a Hamiltonian cycle exists in a given graph (whether directed or undirected). Both problems are NP-complete. Mathematicians and computer scientists alike are familiar with the computational complexity associated with problems referred to as NP-complete. Such problems are included in a group of decision problems known as NP, or nondeterministic polynomial, which have solutions that, once found, can easily be shown to be correct. Although many NP problems can be solved quickly, NP-complete problems cannot, since their complexity grows combinatorially with linear increases in the problem size. These problems are significant because of their relationships to each other: every NP-complete [2] problem can be cast in the form of any other using a polynomial-time algorithm, meaning that an efficient algorithm for one NP-complete problem can be used to solve all others. To check whether a graph has a Hamiltonian Cycle or not is an NP hard problem.

1.4 Literature Reviews

To the best of our knowledge, the quest for good sufficient degree based conditions for Hamiltonian cycles or paths dates back to 1952 when Dirac presented the following theorem, where $\delta(G)$ denotes the degree of the minimum degree vertex of the graph G.

Theorem 1. If G is a simple graph with n vertices, where $n \geq 3$ and $\delta(G) \geq n/2$, then G contains a Hamiltonian cycle.

Later Ore in 1960 presented a highly celebrated result where a lower bound for the degree sum of nonadjacent pairs of vertices was used to force the existence of a Hamiltonian cycle. In particular, Ore proved the following theorem, where d(u) denotes the degree of the vertex u.

Theorem 2(see [3]). Let G be a simple graph with n vertices and u,v distinct nonadjacent vertices of G with $d(u) + d(v) \geq n$. Then,G has a Hamiltonian cycle.

A graph satisfying Ores condition has a diameter of only two , where the diameter of a graph is the longest distance between two vertices. But if a sufficient condition can be derived for a graph with diameter more than two , Hamiltonian path or cycle may be found with fewer edges. With this motivation, Rahman and Kaykobad proposed a sufficient condition to find a Hamiltonian Path in a graph involving the parameter $\delta(u, v)$, which denotes the length of the shortest path between u and v.

Theorem 3(see [4]). Let G=(V,E) be a connected graph with vertices such that for all pairs of distinct nonadjacent vertices $u, v \in$V one has $d(u) + d(v) + \delta(u, v) \geq n + 1$. Then, G has a Hamiltonian path.

In some subsequent literature, the condition, where, $d(u) + d(v) + \delta(u, v) \geq n + 1$, are distinct nonadjacent vertices u,v of a graph having n vertices , is referred to as the Rahman-Kaykobad condition. A number of interesting results were achieved extending and using the Rahman-Kaykobad condition as listed below.

Theorem 4(see [5]). Let G be a 2-connected graph which satisfies the Rahman-Kaykobad condition. If G contains a Hamiltonian path with endpoints at distance 3, then G contains a Hamiltonian cycle.

Theorem 5(see [3]). Let G be a connected graph which satisfies the Rahman-Kaykobad condition. Then, either G contains a Hamiltonian cycle or G belongs to some specific classes of graphs.

Theorem 6(see [6]). Let G be a 2-connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of vertices u and v with $n \geq 2$, then G contains a Hamiltonian cycle, unless is odd and belongs to some specific classes of graphs.

Theorem 7(see [4]). Let G be a 2-connected graph of order $n \geq 6$, which satisfies the Rahman-Kaykobad condition. Then, either G is pancyclic or G belongs to some specific classes of graphs.

Theorem 8. Let G be a connected graph of order n. If max $d(u) + d(v) \geq (n-1)/2$ for each pair of vertices at distance 2, then G is traceable.

1.5 Scope of Thesis

If a graph is viewed as a representation of a finite set of objects and some set of allowed connections between pairs of these objects, then a natural question is can one follow these connections and visit each object in an efficient way? If an efficiency requirement is that each object is visited exactly once, then we are asking if the graph has a Hamiltonian path. If we are required to return to the object from which we began, then we are asking if the graph has a Hamiltonian cycle. This problem was considered as early as 1856 by Thomas Kirkman. If we also assign costs to each connection, then we have the full statement of the famous Traveling Salesman Problem. The Travelling Salesman Problem (TSP) asks the following question: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city? It is an NP-hard problem in combinatorial optimization, important in operations research and theoretical computer science.

TSP is a special case of the travelling purchaser problem. In the theory of computational complexity, the decision version of the TSP (where, given a length L, the task is to decide whether the graph has any tour shorter than L) belongs to the class of NP-complete problems. Thus, it is possible that the worst-case running time for any algorithm for the TSP increases superpolynomially (or perhaps exponentially) with the number of cities. [7]

The problem was first formulated in 1930 and is one of the most intensively studied problems in optimization. It is used as a benchmark for many optimization methods. Even though the problem is computationally difficult, a large number of heuristics and exact methods are known, so that some instances with tens of thousands of cities can be solved completely and even problems with millions of cities can be approximated within a small fraction of 1 percent. The TSP has several applications even in its purest formulation, such as planning, logistics, and the manufacture of microchips. Slightly modified, it appears as a sub-problem in many areas, such as DNA sequencing. In these applications, the concept city represents, for example, customers, soldering points, or DNA fragments, and the concept distance represents travelling times or cost, or a similarity measure between DNA fragments. In many applications, additional constraints such as limited resources or time windows may be imposed. Hypohamiltonian graphs arise in integer programming solutions to the traveling salesman problem: certain kinds of hypohamiltonian graphs define facets of the traveling salesman polytope, a shape defined as the convex hull of the set of possible solutions to the traveling salesman problem, and these facets may be used in cutting-plane methods for solving the problem.Grtschel (1980) observes that the computational complexity of determining whether a graph is hypohamiltonian, although unknown, is likely to be high, making it difficult to find facets of these types except for those defined by small hypohamiltonian graphs; fortunately, the smallest graphs lead to the strongest inequalities for this application.

Concepts closely related to hypohamiltonicity have also been used by Park, Lim andS Kim (2007) to measure the fault tolerance of network topologies for parallel computing. An equivalent formulation in terms of graph theory is: Given a complete weighted graph (where the vertices would represent the cities, the edges would represent the roads, and the weights would be the cost or distance of that road), find a Hamiltonian cycle with the least weight. The requirement of returning to the starting city does not change the computational complexity of the problem, see Hamiltonian path problem. Another related problem is the bottleneck traveling salesman problem (bottleneck TSP): Find a Hamiltonian cycle in a weighted graph with the minimal weight of the weightiest edge. The problem is of considerable practical importance, apart from evident transportation and logistics areas. A classic example is in printed circuit manufacturing: scheduling of a route of the drill machine to drill holes in a PCB. In robotic machining or drilling applications, the ”cities” are parts to machine or holes (of different sizes) to drill, and the ”cost of travel” includes time for retooling the robot (single machine job sequencing problem). [8]

Because of the difficulty of solving the Hamiltonian path and cycle problems on conventional computers, they have also been studied in unconventional models of computing. For instance, Leonard Adleman showed that the Hamiltonian path problem may be solved using a DNA computer. Exploiting the parallelism inherent in chemical reactions, the problem may be solved using a number of chemical reaction steps linear in the number of vertices of the graph; however, it requires a factorial number of DNA molecules to participate in the reaction.

There can have many more applications of Hamiltonian Cycle Problem. Such as:

1.Public transportSS

2.Tour planning

3.Design of microchips

4.Genome sequencing

1.6 Organization of Rest of the Thesis

Chapter 2 discusses about the priliminaries that we needed to write this thesis paper. to understand the problem,we needed to go through the ore’s formulation, two cross over problem and many more.

Chapter 3 discusses the formula we tried to generate by our findings. It is of a three cross over graph. As there is already an idea generated for a two cross over graph.

Chapter 4 is about conclusion and future expansion of our findings.

CHAPTER 2
PRELIMINARIES

2.1 Problem

We consider only simple graphs and hence neither self-loop nor multiedges are present. Suppose we have a graph $G(V, E)$ with n vertices. We sometimes use the notations $V[G] = V$ and $E[G] = E$. Two vertices, $u, v \in V$ are said to be adjacent/neighbours to each other if u, v ; otherwise, they are nonadjacent. The set of neighbours of a vertex u in G is denoted by N(u) . If G' is a subgraph of G and $u[G]$, then $N(u[G'])$ denotes the set of neighbours of u (confined) in G'. Now, $d(u) = N(u)\,|$ and $d[G'] = |Nu[G']\,|$. We use G' to denote the complimentary graph of G; that is, $V\lceil G'\rceil = V - V\lceil G'\rceil$ and $E\lceil G'\rceil = E - E\lceil G'\rceil$. Sometimes, P is referred to as a u,v -path and u and v are referred to as the end vertices or endpoints of P. Also, sometimes we use the notation P to denote the length of P. So, by our definition, $|P| = k - 1$. If we have (x_1, x_k), then the graph $C = (V[P], E')$ such that $E' = E[P] \bigcup (x_1, x_k)$ is called a cycle. In what follows, we only consider simple paths and simple cycles. A path P (cycle C) is called a Hamiltonian path (cycle) if $V[P] = V(V[C] = V)$. Given a path P of G as defined above, assume that $d(x_1[P]) \neq 0$ and $d(x_k[P]) \neq 0$. Two edges $(x_1, x_i), (x_k, x_j) \in E, 1 \leq k$, are said to be crossover edges if and only if $j = i - 1$.

Proof. We easily get a cycle as follows:

C=$\langle x1, x2, x3, ..., xi - 1, xk, xk - 1, ...xi, x1\rangle$

In what follows, we extensively use the following result.

Lemma 9(see [4]). Let $G = (V, E)$ be a connected graph with n vertices and P a longest path in G. If P is contained in a cycle then P is a Hamiltonian path. An independent set of a graph $G = (V, E)$ is a set of vertices V' such that $u, v \in V'$ all pairs of vertices are nonadjacent in G. A graph can be decomposed into independent sets in the sense that the entire vertex set of the graph can be partitioned into pairwise disjoint independent subsets. Such independent subsets are called partite sets or simply parts. A graph is said to be a k-partite graph, if its vertex set can be decomposed into partite sets but not fewer. So, a bipartite graph is a graph that can be decomposed into two partite sets but not fewer. Similarly, a tripartite graph is a graph that can be decomposed into k three partite sets but not fewer. A 1-partite graph is the same as an independent set or an empty graph.

One often writes $G = (A, E)$ to denote a bipartite graph whose partite sets are A and B. If $|A| = |B|$, that is, if the two partite sets have equal cardinality, then G is called a balanced bipartite graph. On the other hand, if $||A| - |B|| \leq 1$, then we say that G is a semibalanced bipartite graph. Note that, by definition, a balanced bipartite graph is also a semibalanced bipartite graph. It is easy to see that, for a bipartite graph G to possess a Hamiltonian path, G must be semibalanced. Similar to the notation used for bipartite graphs, a tripartite graph with partite sets A,B and C may be denoted by $G = (A \bigcup$B$, E)$.

Before presenting some of the relevant conditions of Hamiltonicity existing in the literature, we need to introduce and define some of the notations that will be used throughout this paper. Given a graph $G(V, E)$ and for a vertex $u \in V$, we mean by d(u) the degree of u in G. By $\delta(u, v)$, we denote the length of a shortest path between u and v in G. On the other hand, we denote a Hamiltonian path with end vertices u and v by $H(u, v)$.

Now we are ready to list some of the relevant results available in the literature for the existence of Hamiltonian cycles or paths in graphs.

Theorem 1.1 (Ore [4]). $If d(u) + d(v) \geq$ n for every pair of distinct non-adjacent vertices u and v of G, then G is Hamiltonian.

Theorem 1.2 (Rahman and Kaykobad [5]). Let $G = (V, E)$ be a connected graph with n vertices and P be a longest path in G having length k and with end vertices u and v. Then the following statements must hold:

(a) Either $(u, v) > 1$ or P is a Hamiltonian path contained in a Hamiltonian cycle.

(b) If $(u, v) \geq$ 3 then $d(u) + d(v) - \delta(u, v) + 2$

(c) If $\delta(u, v) = 2$, then either $d(u) + d(v) \leq k$ or P is a Hamiltonian path and there is a Hamiltonian cycle.

Theorem 1.3 (Rahman and Kaykobad [5]). Let $G = (V, E)$ be a connected graph with n vertices such that for all pairs of distinct non-adjacent verticesu,$v \in V$ we have $d(u) + d(v) + \delta(u, v) \geq n + 1$. Then G has a Hamiltonian path.

But with the inclusion of the length of the shortest path in Theorem 1.3, there is a possibility of sparser graphs qualifying for containing Hamiltonian paths.

Lemma 1.1 Let G be a simple graph with n vertices and u v, be distinct non-adjacent vertices of G with $d(u) + d(v) \geq n$. Then $\delta(u, v) = 2$ This Lemma 1.1 along with Lemma 3.2 in [5] also implies the validity of Ores theorem.

2.2 Sufficient conditions for Hamiltonian city

A graph is said to be Hamiltonian if it contains a spanning cycle. The spanning cycle is called a Hamiltonian cycle of G, and G is said to be a Hamiltonian graph . A Hamiltonian path is a path that contains all the nodes in V(G) but does not return to the node in which it began. No characterization of Hamiltonian graphs exists, yet there are many sufficient conditions. We begin our investigation of sufficient conditions for Hamiltonicity with two early results. The first is due to Dirac, and the second is a result of Ore. Both results consider this intuitive fact: the more edges a graph has, the more likely it is that a Hamiltonian cycle will exist. Many sources on Hamiltonian theory treat Ores Theorem as the main result that began much of the study of Hamiltonian graphs, and Diracs result a corollary of that result. Dirac's result actually preceded it, however, and in keeping with the historical intent of this paper, we will begin with him.

Theorem 1.1: *(Dirac, 1952 [9, 10]): If G is a graph of order n* $\geq$ *3 such that d* $\geq$ *n/2, then G is Hamiltonian.*

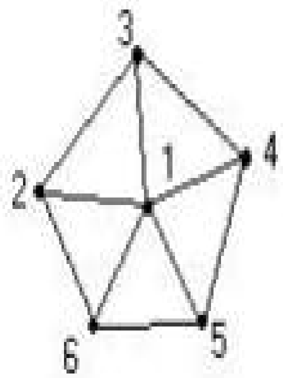

Figure 2.1: Hamiltonian graph

As an illustration of Diracs Theorem, consider the wheel on six nodes,W_6 in the above figure. In this graph, $\delta = 3 \geq (\frac{6}{2})$, so it is Hamiltonian. Traversing the nodes in numerical order 1-6 and back to 1 yields a Hamiltonian cycle.

Theorem 1.2: *(Ore, 1960, [11]): If G is a graph of order n* ≥ 3 *such that for all distinct nonadjacent pairs of nodes u and v,* $deg(u) + deg(v) \geq n$*, then G is Hamiltonian.*

The wheel, W_6, also satisfies Ore's Theorem. The sum of the degrees of nonadjacent nodes

(i.e., deg(2) + deg (5), or deg(3) + deg (6), etc.) is always 6, which is the order of the graph. Before we discuss the results of Nash-Williams and Chvatal and Erdos, we must first define the notions of connectivity and independence.
The connectivity $\kappa = \kappa(G)$ of a graph G is the minimum number of nodes whose removal results in a disconnected graph. For $\kappa \geq$ k, we say that G is k-connected.We will be concerned with 2-connected graphs, that is to say that the removal of fewer than 2 nodes will not disconnect the graph. For $\kappa = k$, we say that G is strictly k-connected. For clarification purposes, consider the following. Let G be any simple graph, $\kappa = 3$. Then G is 3-connected, 2-connected, and strictly 3-connected.
A set of nodes in G is independent if no two of them are adjacent. The largest number of nodes in such a set is called the independence number of G, and is denoted by b. The following result by Nash-Williams builds upon the two previous results by adding the condition that G be 2-connected and using the notion of independence.

Theorem 1.3: *(Nash-Williams, 1971, [12]): Let G be a 2-connected graph of order n with* $\delta(G) \geq max\{(n+2)/3, b\}$. *Then G is Hamiltonian.*

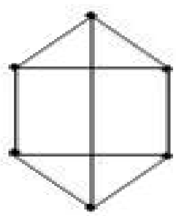

Figure 2.2: Hamiltonian graph

The graph in Figure 2.2 demonstrates the Nash-Williams result. In this 2- connected graph on six nodes, δ = 3, β = 2, and $\delta \geq$ max $\left\{\frac{6+2}{3}, 2\right\}$, implying Hamiltonicity.
In the same paper, Nash-Williams presents another very useful result. Note that a cycle C is a dominating cycle in G if V(G - C) forms an independent set.

Theorem 1.4: *(Nash-Williams, 1971, [12]): Let G be a 2-connected graph on n vertices with* $\delta \geq$ *(n+2)/3. Then every longest cycle is a dominating cycle.*

Another sufficient condition uses the notion of a forbidden subgraph, i.e., a graph that cannot be a subgraph of any graph under consideration. A subgraph of a graph G is a graph having all of its nodes and edges in G. The following result by Goodman and Hedetniemi introduces the connection between certain subgraphs and the existence of Hamiltonian cycles. A bipartite graph G is a graph whose node set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2. If G contains every possible edge joining V_1 and V_2, then G is a complete bipartite graph. If V_1 and V_2 have m and n nodes, we write G = Km,n (see Figure 2.3)

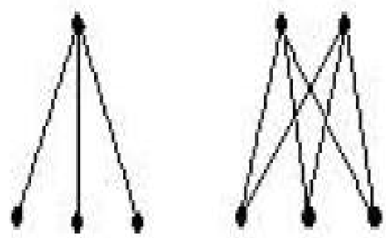

Figure 2.3: $k_{1,3}$ and $k_{2,3}$

Goodman and Hedetniemi connected $K_{1,3}$, $K_{1,3}$ + x-free graphs and Hamiltonicity in 1974. A $\{K_{1,3}$, $K_{1,3}$ + x$\}$-free graph is a graph that does not contain a $K_{1,3}$ or a $K_{1,3}$ + x (see Figure 2.4) as an induced subgraph. (i.e., the maximal subgraph of G with a given node set S of V(G).)

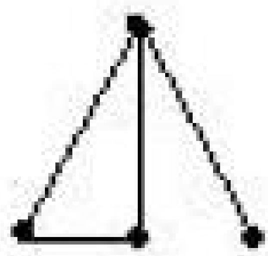

Figure 2.4: $k_{1,3}$ + x

Theorem 1.5: *(Goodman and Hedefniemi, 1974, [13]): If G is a 2-connected $\{K_{1,3}$, $K_{1,3}$ + x $\}$-free graph, then G is Hamiltonian.*

The wheel, W_6, in Figure 2.1, is an example of a graph that is $\{K_{1,3}$, $K_{1,3}$ + x$\}$-free. The subgraph formed by node 1 and any three consecutive nodes on the cycle is $K_{1,3}$ plus 2 edges. A year after Nash-Williamss result, Chvatal and Erdos proved a sufficient condition linking the ideas of connectivity and independence.

Theorem 1.6: *(Chvatal and Erdos, 1972, [10]): Every graph G with n $\geq$ 3 and $\kappa \geq$ b has a Hamiltonian cycle.*

Chvatal and Erdoss result can be demonstrated by the graph in Figure 2.5. In this graph, κ=2 and β=2.

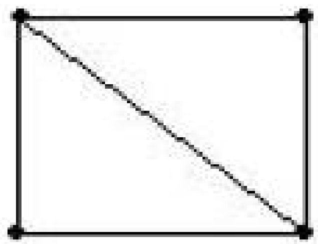

Figure 2.5: Hamiltonian graph

Theorem 1.7: *(Bauer and Schmeichel, 1991, [14]): Let G be a 1-tough graph of order n with δ(G) $\geq$ (n+ κ - 2)/3. Then G is Hamiltonian*
Theorem 1.7 is best possible if κ = 2 (see Figure 2.6).

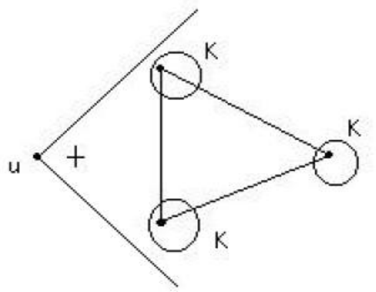

Figure 2.6: Hamiltonian graph

Figure 2.6 is comprised of 3 K_r, r $\geq$ 2, joined with a single node u. In this case, G is a 2- connected, 1-tough graph and δ = *r* = *(n+κ-3)/3* (i.e., δ ¡ *(n+ κ - 2)/3)*. By relaxing the minimum degree requirements, we lose Hamiltonicity.
Fan later introduced distance as a contributing factor for Hamiltonicity. The distance, d(u,v), between two nodes u and v is the length of the shortest path joining them.

CHAPTER 3
NEW CONDITION

From the Ore's theorem and the prelimineries mentioned we get to know that when there is cross edge the graph is a hamiltonian. So it is necessary to know that if there is any cross edge in the graph so to be sure that the graph is hamiltonian. Until now we got to know the conditions of having two cross edge in the graph. Here we will introduce the conditions of three cross edge in a graph. Before going to the main results we will give an overview of ores theorem and two crossoveres.

3.1 Overview of Ores theorem

Formal statement is : Let G be a (finite and simple) graph with $n \geq 3$ vertices. We denote by deg v the degree of a vertex v in G, i.e. the number of incident edges in G to v. Then, Ore's theorem states that if $deg(v) + deg(w) \geq n$ for every pair of non-adjacent vertices v and w of G (*) then G is Hamiltonian.

proof: It is equivalent to show that every non-Hamiltonian graph G does not obey condition (*). Accordingly, let G be a graph on $n3$ vertices that is not Hamiltonian, and let H be formed from G by adding edges one at a time that do not create a Hamiltonian cycle, until no more edges can be added. Let x and y be any two non-adjacent vertices in H. Then adding edge xy to H would create at least one new Hamiltonian cycle, and the edges other than xy in such a cycle must form a Hamiltonian path $v_1v_2...v_n$ in H with $x = v_1$ and $y = v_n$. For each index i in the range $2 \leq i \leq n$, consider the two possible edges in H from v_1 to v_i and from $v(i1)$ to v_n. At most one of these two edges can be present in H, for otherwise the cycle $v_1v_2...v(i1)v_nv(n1)...v_i$ would be a Hamiltonian cycle. Thus, the total number of edges incident to either v_1 or v_n is at most equal to the number of choices of i, which is $n1$. Therefore, H does not obey property (*), which requires that this total number of edges $deg(v_1) + deg(v_n)$ be greater than or equal to n. Since the vertex degrees in G are at most equal to the degrees in H, it follows that G also does not obey property (*).

3.2 Two Crossover

As we introducing three crossovers it is important to understand two crossover first. When two edges approches from very most corner vertices and one falls on a vertex immediate to the vertex where another falls two crossover formed. And from the Ores theorem it is proved that when $d(u) + d(v) \geq n$ two crossover formed.

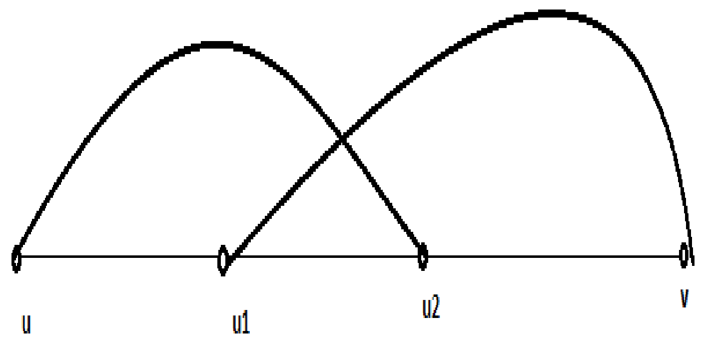

Figure 3.1: Two Crossover Graph(a)

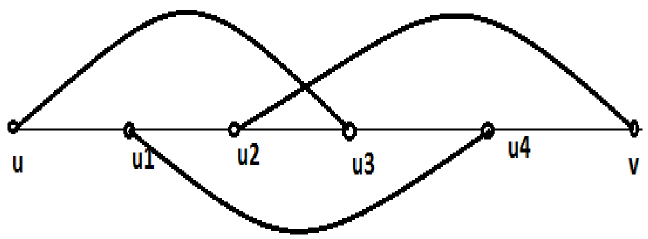

Figure 3.2: Two Crossover Graph(b)

3.3 Study on three crossover for a graph to be hamiltonian

It is mentioned earlier that whenever there is cross edge the graph will form a cycle covering all the vertices and will be hamiltonian. Here we think about the concept of three crossover and thereby possible conditions. When two edges from two corner most vertices incident on two vertices and another edge arises from a vertex which is immidiate previous to the vertex

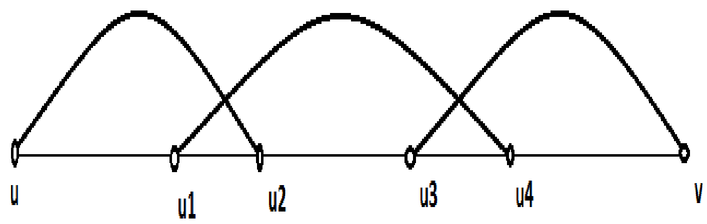

Figure 3.3: Three Crossover Graph (a)

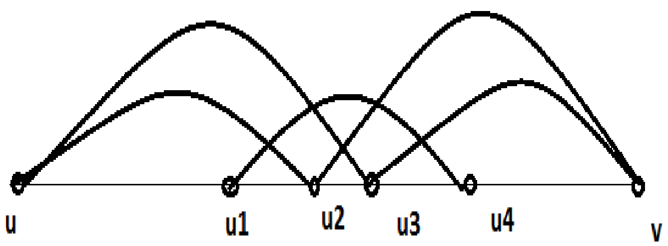

Figure 3.4: Three Crossover Graph(b)

where other end of corner most edge incidents and falls on a vertex which is immidiate previous to the vertex where other end of another corner most edge incidents.

In a graph there may remain three crossovers but no two crossovers. The number of edges needed to be three crossover can be less then the number of edges needed for being two crossover. In that case it will be more efficient way to prove a graph to be hamiltonian.

If we consider that total number of midcrosses is X. Degree of the corner most vertex u is d(u) and degree of another corner vertex v is d(v). And total number of vertices is n. Then the possible number of edges for the graph can be *e=(d(u)-1)+(d(v)-1)+X+n-1* Here one edge from both u and v has been omitted because this edge has already been calculated by notation $n - 1$. Now we comes to calculation of X ie. midcrosses.

In calculating midcrosses of the vertices it affects the counting of degree of u and degree of v. Because we count the edge of u i.e left corner repeatedly each time whenever the edges crosses that edge. Same case goes for the edges of rightmost vertex v. So it is seen that left and right crosses is counted equal to the number of X. So here we have to estimate this

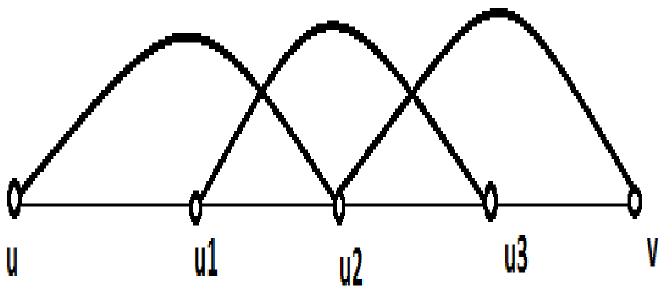

Figure 3.5: Three Crossover Graph(c)

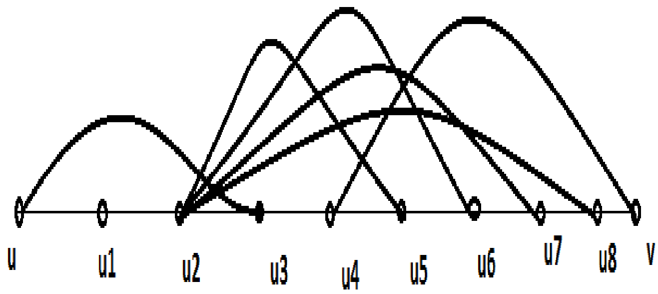

Figure 3.6: Three Crossover Graph(d)

repetition to get more accurate result. Let P be the number of edges repeated equal to X. So *P=[(*d_i*-2-*d_ir*)]*. Here d_ir is the number of edges comes from the right side in forward direction. In the same way for the right cross or end cross *Q=[(*d_i*-2-*d_il*)]* where d_i is the number of edges comes from the left side in backward direction.And this is also equal to X. So we can claim that *P+Q=2X*. If that then *X=(P+Q)/2*. But this condition still does not estimate the repeatition occurs. There are lot more repeatition of edges so does not give the actual number of edges.

So we thought in another way-counted the number of midcrosses.we consider midcrosses arising from the vertex other than the left and right vertex. Let the number of vertex covered by the midcrosses is X. One end of the vertices fall on left side and another end on right side. And we consider that the number of vertices on right side is floor of $X/2$ whereas the number of vertices on the left side is $X - X/2$ which also apparently $X/2$.

Edges arisng from the left vertex falls on $(X/2)/2$ ie $X/4$ vertices of the left. And edges arises from the right vertex falls on $X/4$ vertices of the right but this right $X/4$ vertices

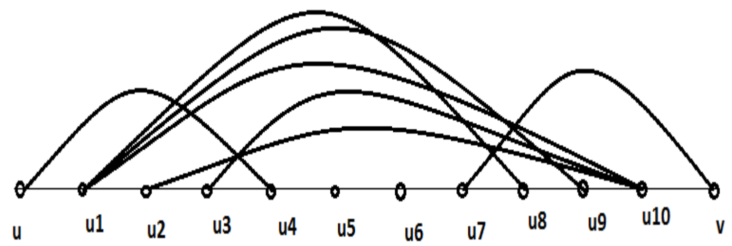

Figure 3.7: Three Crossover Graph(e)

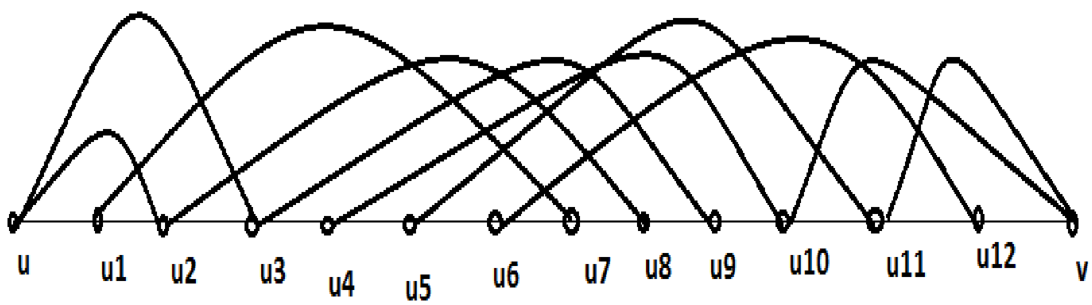

Figure 3.8: Three Crossover Graph(f)

are different from the other end vertices of the edges arises from the vertex $X/4$ of the left sides. So that no cycle form in this way. But now if we add any edges from the left or right vertex there must be three crossover. So the number of edges of left and right vertex are $X/2$. So if this number increase by one there must be three cross over. So we can tell that when the degree of left and right vertex are greater than or equal to $X/2 + 1$ there will exist three crossovers. It is easily said that this X is less than the total number of vertices n. That means it is not necessary to have two crossover when there are three crossover and with the condition that there remain lesser number of edges then two crossover, three crossover can occur. It can be said that $d(u) + d(v) \geq min(n, X/2 + 1)$. In the worst case it can be equal to n. Now the challenge is to find the value of X which we are still working on.

CHAPTER 4
CONCLUSION AND RECOMMENDATION

4.1 Conclusion

We have seen that the Hamiltonian Cycle problem is currently of interest in the areas of graph theory, algorithms, and complexity theory. The Look-Ahead Theorem gives us a tool for also taking it into the area of proof complexity. The goal of this research would be to see how powerful this proof system is. In chapter 1 and 2 we have seen the history and necessary conditions for a grpah to be hamiltonian. Here we knew that a grpah is hamiltonian if it contains two cross-over edge. In chapter 3 we have tried to find a new condition for a graph to be hamiltonian when it contains three-crossover edge. When the graph contains three cross-over edge then we tried to delete a cross-over edge and make the graph to be hamiltonian for two cross-over edge. Then we tried to calculate the minimum number of deletion edge. After that we calculate the minimum number of edge for a graph to be hamiltonian when it contains three cross-over edge. Cross-overs are very effective in constructing a long path in a graph. The condition is based on theoritical technique to prove graphs hamiltonian.

4.2 Recommendation

We are trying to find a condition for a graph to be hamiltonian when it contains three cross-over edge. It can be found to a simple condition for N cross-over edge for future expansion. If we can establish the grpah to be hamiltonian for N cross-over then we can easily implement it to our practical life with a minimun number of distance cover.

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