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Mathematical Modelling of Vehicle Drifting

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ABSTRACT

A mathematical model and condition for drifting of vehicles are presented in this paper. Employing the condition for possible steady-state drifting, the mathematical model of a vehicle with lateral weight lift during turning and drifting as well as adopting a combined tyre force model enables to reduce the number of equations of motion to a set of nonlinear coupled algebraic equations. The solution of the equations are the longitudinal and lateral components of the velocity vector of the vehicle at its mass centre and the vehicle's yaw rate. The numerical values of the variables are associated with an equilibrium at which the vehicle drifts steadily. The equilibrium point should be analysed for stability by examining for any small disturbance should disappear. The procedure applied to a nominal vehicle indicates that an equilibrium point exists for every given value of the steering angle as the input. Also, it is shown that the equilibrium point is unstable. Hence, to keep the vehicle at the associated steady-state drifting, the value of the yaw rate must be kept constant.

1. INTRODUCTION

Race-car drivers wish to keep the speed as high as possible in corners. Such a manoeuvre may go into drift when the vehicle slides laterally on the road. Drift may also be controlled intentionally when the inwards and outwards motion of the vehicle are balanced while the vehicle is moving the desired path. Generally, race cars are made to be rear-wheel-drive to improve their acceleration performance. Considering tyre force dynamics, increased longitudinal slip at rear tyres will cause a drop of lateral force and makes a larger side-slip angle at the rear wheels (Jazar, 2019). Therefore, the driver needs to adjust the steering and torque input to the vehicle to keep the vehicle on track. To prevent the vehicle from spinning, steering the vehicle opposite to turning direction may also be needed to keep the vehicle on track. The sensitivity of motion of such vehicles to inputs is high (Milani *et al.*, 2019, Voser *et al.*, 2010).

Steady-state drifting is an unstable manoeuvre and hard to explain mathematically. The topic has been under investigation by several researchers, (Bobier-Tiu *et al.*, 2019). Edelmann and Plöchl (2009) showed that the equilibrium point corresponding to the drifting of a four-wheel vehicle is unstable by linearizing the equations around the equilibrium point. Velenis, Frazzoli, and Tsiotras (2010) analysed a vehicle bicycle model by considering longitudinal load transfer and employing the magic formula tyre model. They applied a control scheme to stabilize the equilibrium point by adjusting steering and torque inputs. Velenis *et al.* (2011) proposed an LQR control to adjust the steering angle and rear wheels' angular velocity to keep their vehicle model drifting. An iterative method has also been used by Chaichaowarat and Wannasuphoprasit to derive steady-state drifting of a bicycle model (Chaichaowarat & Wannasuphoprasit, 2013). There are also several attempts to apply different control strategies to keep steady-state drifting on bicycle models (Hindiyeh & Gerdes, 2009, Voser, Hindiyeh, & Gerdes, 2010, Hindiyeh & Gerdes, 2014). Figure 1 illustrates a bicycle vehicle model at steady state drifting.

In this paper, a mathematical model for identifying drifting manoeuvres in steady-state condition and controlling the vehicle at those conditions are presented. Several criteria have been developed to evaluate drifting manoeuvres (Abdulrahim, 2006). Quantitative methods have also been introduced by measuring vehicle body side-slip angle at its mass centre (Abdulrahim 2006, Hindiyeh & Gerdes, 2014). Defining drifting has also been done as a large sideslip manoeuvres in steady state (Edelmann & Plöchl, 2009, Velenis, Frazzoli, & Tsiotras, 2010). Manoeuvres with large sideslip, counter-steering, and saturation of lateral tyre forces at the rear are also identified as drifting (Hindiyeh & Gerdes, 2009). Drifting may also happen unfavourably at high speeds when a driver expects to negotiate the vehicle turning in a certain direction, but the vehicle will not follow the driver's command when tyres are at large slips. Drifting is usually manoeuvring with side-slip angles over the limit and opposite steering, (Tavernini *et al.*, 2013, Shi *et al.*, 2017). This study develops a kinematic condition for feasible counter steering as a mathematical expression to include kinematic vehicle variables. A general definition that is not only applied to a specific instant of driving but also accounts for all principal forces and moments during the manoeuvre. The vehicle will turn about the centre of rotation with a smaller radius during drifting generating large centripetal acceleration. The resultant difference in the longitudinal force of left and right driving wheels generates a yaw moment about the mass centre of the vehicle that will be turning the vehicle, (Hindiyeh & Gerdes, 2009, Voser, Hindiyeh, & Gerdes, 2010, Hindiyeh & Gerdes, 2014).

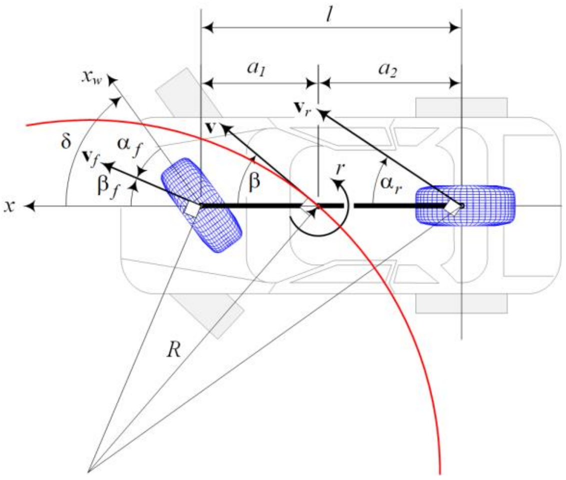

Figure 1: A two-wheel vehicle model at steady state drifting

To analyse drifting manoeuvres, the suitability of the vehicle model is of importance while the level of simplifications must be kept reasonable such that the model captures the key dynamic behaviours of the vehicle during drifting. We use a rear-wheel-drive vehicle and assume no positive or negative longitudinal force is present at the front tyres. Therefore, we may integrate front wheels into a single wheel without losing accuracy. However, we keep the rear wheels separate to include weight transfer during turning. Figure 2 illustrates such a mathematical model.

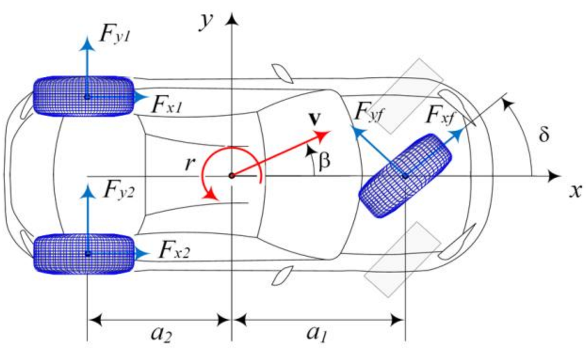

Figure 2: Three-Wheel Vehicle Model

2. MATHEMATICAL VEHICLE MODEL

The general forms of equations of motion for a rigid body in planar motion in body coordinate are (Jazar, 2019):

$$\Sigma F_x = m\dot{v}_x - mrv_y \tag{1}$$

$$\Sigma F_y = m\dot{v}_y + mrv_x \tag{2}$$

$$\Sigma M_z = I_z\dot{r} \tag{3}$$

where, v_x, v_y, and r are longitudinal, lateral, and yaw velocities of the mass centre, respectively. As shown in Figure 2, the total longitudinal and lateral forces may be summarized as:

$$\Sigma F_x = F_{x1} + F_{x2} + F_{xf}\cos\delta - F_{yf}\sin\delta$$
$$= F_{xr} + F_{xf}\cos\delta - F_{yf}\sin\delta \tag{4}$$

$$\Sigma F_y = F_{y1} + F_{y2} + F_{yf}\cos\delta + F_{xf}\sin\delta$$
$$= F_{yr} + F_{yf}\cos\delta + F_{xf}\sin\delta \tag{5}$$

$$\Sigma M_z = \begin{matrix}(F_{yf}\cos\delta + F_{xf}\sin\delta)a_1 - (F_{y1} + F_{y2})a_2 \\ +(F_{x2} - F_{x1})\frac{w}{2}\end{matrix}$$
$$= (F_{yf}\cos\delta + F_{xf}\sin\delta)a_1 - F_{yr}a_2 + \Delta F_{xr}\frac{w}{2} \tag{6}$$

Steer angle δ and side-slip angle β are indicated in the figure, dimensions a_1 and a_2 are distances of the front and rear axles from vehicle mass centre, respectively, and w shows the rear track width. We adopt the combined elliptical tyre forces (Jazar, 2019). The longitudinal and lateral tyre forces are:

$$F_x = F_z C_s S(s - s_s)\sqrt{1 - C_{s\alpha}\left(\frac{S(\alpha-\alpha_s)}{\alpha_s}\right)^2} \tag{7}$$

$$F_y = -F_z C_\alpha S(\alpha - \alpha_s)\sqrt{1 - C_{\alpha s}\left(\frac{S(s-s_s)}{s_s}\right)^2} \tag{8}$$

Parameters C_s and C_α are the longitudinal and lateral tyre stiffnesses, while $C_{s\alpha}$ and $C_{\alpha s}$ are longitudinal and lateral drop factors, and function S is the saturation function which limits the magnitude of the input variable to a maximum. Parameters s_s and α_s are the saturation slip values at which the tyre forces are assumed to be saturated when the slip value reaches these certain limits. The combined tyre force indicates that the introduction of a secondary slip causes a drop in the current tyre force until both slips are saturated.

The lateral load transfer at the rear of the car is estimated as follows (Jazar, 2019):

$$\Delta F_{zr} = F_{z2} - F_{z1} = F_{zr}\frac{2h(\dot{v}_y + v_x r)}{wg}$$
$$= m\left(\frac{a_1 g}{a_1+a_2} + \frac{h(\dot{v}_x - rv_y)}{a_1+a_2}\right).\frac{2h(\dot{v}_y + v_x r)}{wg} \tag{9}$$

While vertical loads under the front and rear axles are being estimated by the following equations (Jazar, 2019):

$$F_{zf} = m\left(\frac{a_2 g}{a_1+a_2} - \frac{h(\dot{v}_x - rv_y)}{a_1+a_2}\right) \tag{10}$$

$$F_{zr} = m\left(\frac{a_2 g}{a_1+a_2} + \frac{h(\dot{v}_x - rv_y)}{a_1+a_2}\right) \tag{11}$$

where h is the mass centre height of the vehicle from the ground.

3. MATHEMATICAL DEFINITION OF DRIFTING

We begin by assuming a steady-state drifting manoeuvre to come up with a kinematic condition for the drifting vehicle. At the assumed steady-state turning condition, the equation of motion in yaw for the three-wheel planar vehicle in steady-state must be:

$$\Sigma M_z = 0 \tag{12}$$

A large torque at rear wheels reduces lateral tyre force at the rear and creates a combined-slip condition. The rear tyres will spin at this condition as the longitudinal slip saturates while large tyre side-slip angles happen, and tyres will be side-slip saturate. Hence, it is assumed both rear tyres are laterally and longitudinally saturated. The total lateral tyre force at the rear will then be:

$$F_{yr} = \sqrt{1 - C_{\alpha s}}\mu_y F_{zr} \tag{13}$$

Combining (13) and (12) and ignoring derivative terms at steady state makes the yaw moment equation become:

$$\Sigma M_z = F_{yf} a_1 - \sqrt{1 - C_{\alpha s}}\mu_y F_{zr} a_2 + \frac{h\mu_x v_x r}{g} F_{zr} = 0 \tag{14}$$

The equation is solved for F_{yf} and substitute a nominal range of vehicle's parameters, to conclude that the total lateral force at the front must be positive to satisfy the yaw motion in equilibrium:

$$F_{yf} = F_{zr} \frac{\sqrt{1-C_{\alpha s}}\mu_y a_2 - h\mu_x v_x r/g}{a_1} \tag{15}$$

$$g = 9.81 \text{m/s}^2, \quad C_{\alpha s} \approx 0.5, \quad \mu_x \approx \mu_y \approx 0.75,$$

$$a_2 > 1m, \quad h < 1m, \quad v_x r < 0.5g$$

$$F_{yf} > 0.15 \frac{F_{zr}}{a_1} > 0 \tag{16}$$

Having a positive lateral force at the front tyre makes the tyre side-slip angle in a left-hand turn to be negative. The side-slip angle of the vehicle body at centre of the front wheel is shown by β_f, tyre side-slip angle by α_f, and the steering angle by δ and they are related by (Jazar, 2019):

$$\alpha_f = \beta_f - \delta = arctan \frac{v_{yf}}{v_{xf}} - \delta \tag{17}$$

$$\beta_f - \delta < 0 \rightarrow \beta_f < \delta \tag{18}$$

In case of a negative steer angle to cause left-hand turning of the vehicle:

$$\beta_f < \delta < 0 \tag{19}$$

$$r > 0 \tag{20}$$

Employing (19) and (20), the kinematic condition for drifting may now be defined as:

$$r\beta_f < 0 \tag{21}$$

Therefore, it is proposed that whenever the yaw velocity and body side-slip angle at the front wheel have opposite signs, the vehicle is drifting. The condition also provides us with an equation to associate a value to the instantaneous drifting at any time. Such value suggests that a drift meter can be introduced, and the future vehicle can be equipped with a warning system to inform the driver when the vehicle is approaching a drifting condition. To evaluate a manoeuvre in terms of drifting, we integrate the expression over time of the manoeuvre.

4. PHASE PLANE ANALYSIS AND DRIFTING EQUILIBRIUM POINT

As assumed, the steady-state condition for drifting must be associated with an equilibrium condition and it might be shown geometrically in suitable state space. To investigate this phenomenon, the time derivative terms are removed as the state of the system should not change at the equilibrium point. Therefore, the equations of motion in equilibrium will be:

$$\Sigma F_x = -mrv_y \tag{22}$$

$$\Sigma F_y = mrv_x \tag{23}$$

$$\Sigma M_z = 0 \tag{24}$$

Substituting the tyre force model (7) and (8), total longitudinal and lateral forces will be written as:

$$\Sigma F_x = \sqrt{1 - C_{s\alpha}}\ \mu_x m \left(\frac{a_1 g}{a_1+a_2} - \frac{hrv_y}{a_1+a_2}\right)$$
$$+m\left(\frac{a_2 g}{a_1+a_2} + \frac{hrv_y}{a_1+a_2}\right) C_{\alpha f}\left[arctan\left(\frac{a_1 r + v_y}{v_x}\right) - \delta\right] sin\,\delta \tag{25}$$

$$\Sigma F_y = \sqrt{1 - C_{\alpha s}}\ \mu_y m \left(\frac{a_1 g}{a_1+a_2} - \frac{hrv_y}{a_1+a_2}\right)$$
$$-m\left(\frac{a_2 g}{a_1+a_2} + \frac{hrv_y}{a_1+a_2}\right) C_{\alpha f}\left[arctan\left(\frac{ra_1 + v_y}{v_x}\right) - \delta\right] cos\,\delta \tag{26}$$

$$\Sigma M_z = \frac{hm\ \mu_x v_x r\left(\frac{a_1 g}{a_1+a_2} - \frac{hrv_y}{a_1+a_2}\right)}{g}$$
$$-\sqrt{1 - C_{\alpha s}}\ \mu_y m\left(\frac{a_1 g}{a_1+a_2} - \frac{hrv_y}{a_1+a_2}\right) a_2$$
$$-m\left(\frac{a_2 g}{a_1+a_2} + \frac{hrv_y}{a_1+a_2}\right) C_{\alpha f}\left[arctan\left(\frac{ra_1 + v_y}{v_x}\right) - \delta\right] cos\,\delta\, a_1 \tag{27}$$

Substituting these force systems in equations of motion leads us to a set of algebraic equations to be solved for the vehicle velocity components v_x, v_y and r for a given value of steer angle as input. There is no closed-form solution for the set of equations and hence, they must be solved numerically for given nominal vehicle parameters to determine the expected equilibrium points. The nominal values are shown in Table 1 (Milani *et al*., 2019).

Solving Equations (25)-(27) for the velocities the values of v_x, v_y and r is found when the vehicle is drifting steadily for given values of steer angle. The values shown in Table 2 are the solutions of the equations.

Table 1
Nominal values of vehicle parameters

m	h	$a1$	$a2$
1600 kg	0.9 m	1.35 m	1.5 m
Cs	$C\alpha$	$Cs\alpha$	$C\alpha s$
7.5	8.5	0.5	0.5
w	Iz	Rw	Iw
1.58 m	2000 $kg.m2$	0.35 m	1 $kg.m2$
ss	αs	μx	μy
0.1	0.09 $[rad]$	0.75	0.75

Table 2
Values of v_x, v_y, r and δ when the vehicle is drifted steadily for given values of steer angle

$r\ (rad/s)$	$vx\ (m/s)$	$vy\ (m/s)$	$\delta\ (rad)$
0.844756776	6.080326946	−5.027221269	−0.5
1.023066746	4.960105873	−3.860670023	−0.4
1.152547811	4.353764384	−3.192852462	−0.3
1.254922383	3.956398895	−2.73254504	−0.2
1.340414747	3.666314032	−2.380578459	−0.1
1.414736369	3.438840107	−2.092329884	0

To examine the stability of the equilibrium points, the set of equations of motion around each equilibrium point may be linearized and the associated eigenvalues are calculated as

$$\dot{v}_x = \frac{\partial f_1(v_x,v_y,r)}{\partial v_x} v_x + \frac{\partial f_1(v_x,v_y,r)}{\partial v_y} v_y + \frac{\partial f_1(v_x,v_y,r)}{\partial r} r + g_1(\delta) \quad (28)$$

$$\dot{v}_y = \frac{\partial f_2(v_x,v_y,r)}{\partial v_x} v_x + \frac{\partial f_2(v_x,v_y,r)}{\partial v_y} v_y + \frac{\partial f_2(v_x,v_y,r)}{\partial r} r + g_2(\delta) \quad (29)$$

$$\dot{r} = \frac{\partial f_3(v_x,v_y,r)}{\partial v_x} v_x + \frac{\partial f_3(v_x,v_y,r)}{\partial v_y} v_y + \frac{\partial f_3(v_x,v_y,r)}{\partial r} r + g_3(\delta) \quad (30)$$

The linearized equations for $\delta = -0.5\,rad$, and their associated eigenvalues will be:

$$\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{r} \end{bmatrix} = [A]_{3\times 3} \begin{bmatrix} v_x \\ v_y \\ r \end{bmatrix} + \begin{bmatrix} g_1(\delta) \\ g_2(\delta) \\ g_3(\delta) \end{bmatrix} = \begin{bmatrix} -1859.94 & -1664.73 & -11336.35 \\ -4756.21 & -5333.39 & -16874.67 \\ -2.02 & -2.02 & -4.29 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ r \end{bmatrix} + \begin{bmatrix} g_1(\delta) \\ g_2(\delta) \\ g_3(\delta) \end{bmatrix} \quad (31)$$

$$\mathrm{eig}([A]) = \begin{bmatrix} -6911.51 \\ -291.71 \\ 5.59 \end{bmatrix} \quad (32)$$

In this case, two of the three eigenvalues have negative real parts indicating stability with respect to two of the system variables. However, one eigenvalue has a positive real part indicating instability of the linear system with respect to one of the variables. Figure 3 shows the phase portrait of the system variables. The phase portraits show that the equilibrium point is stable in the large for v_x, v_y when r is kept constant, but unstable when r varies. Therefore, keeping yaw velocity constant by means of a feedback control, the equilibrium drifting point is stable, and it can be expected a steady-state drifting motion as is shown in Figures 4 and 5.

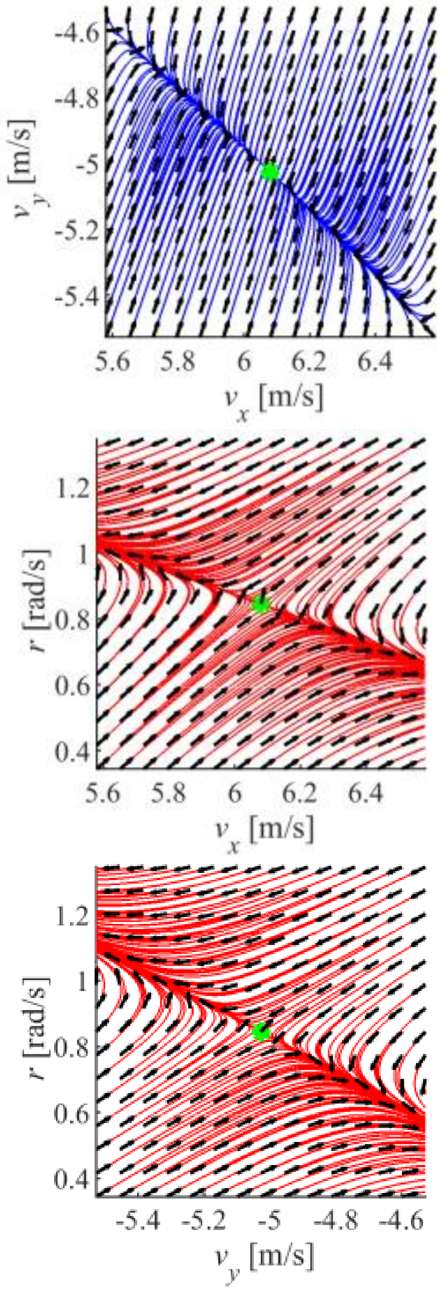

Figure 3: Phase Portraits of the Three-Wheel Model at Equilibrium Point

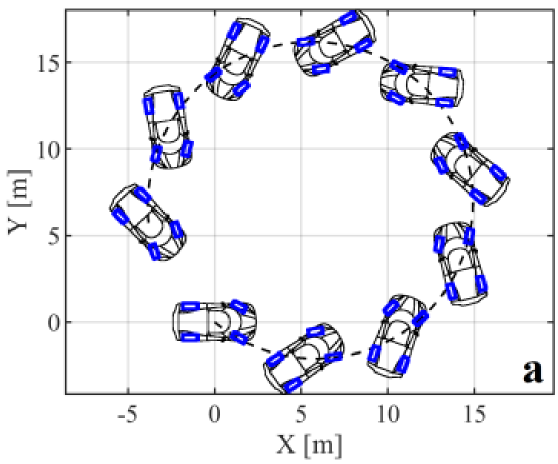

Figure 4: Steady-state drifting condition of a nominated vehicle

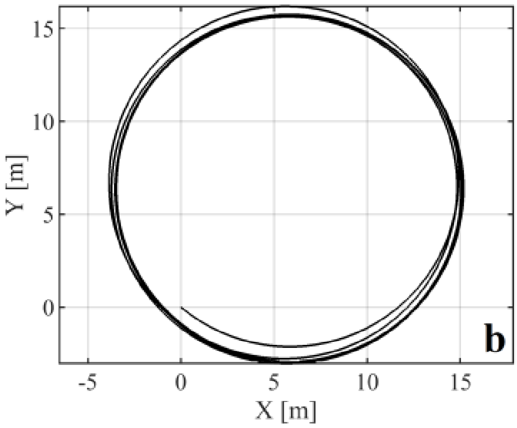

Figure 5: Nominated vehicle's path in the global coordinate frame

5. CONCLUSIONS

This paper introduces a condition for steady-state drifting condition of the vehicle by investigating equilibrium points of the equations of motion of vehicles. The lateral and longitudinal load transfer has been included. The tyre model also was capable to consider the combined slips as well as combined saturation of tyre forces. The equations of motion at steady state condition have been reduced to a set of nonlinear coupled algebraic equations with longitudinal and lateral velocities, and yaw rate as unknown. The equations may be loaded by parameters of any given vehicle to be solved for the unknowns at a given steer angle input. If there is any solution for the variables, then they will indicate equilibrium points of drifting. The stability analysis by employing the eigenvalue method of the linearized equations around the equilibrium point determines the stability of the equilibrium point. A sample example for a set of nominated numerical values as a fixed steer angle indicated the existence of an equilibrium point suggesting the existence of steady-state drifting. Stability analysis of the equilibrium point suggest that it is possible to keep the vehicle at a steady-state drifting condition when keeping the value of the yaw rate constant by means of a control system.

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REFERENCES

Abdulrahim, M. (2006). On the dynamics of automobile drifting. *Technical Paper 2006-01-1019*, *SAE 2006 World Congress & Exhibition*, https://doi.org/10.4271/2006-01-1019

Bobier-Tiu, C. G., Beal, C. E., Kegelman, J. C., Hindiyeh, R. Y., & Gerdes, J. C. (2019). Vehicle control synthesis using phase portraits of planar dynamics, *Vehicle System Dynamics*, 57(9), 1318-1337

Edelmann, J., & Plöchl, M. (2009). Handling characteristics and stability of the steady-state powerslide motion of an automobile. *Regular and Chaotic Dynamics*, 14(6), 682, https://doi.org/10.1134/S1560354709060069

Hindiyeh, R. Y., & Gerdes, J. C. (2009). Equilibrium analysis of drifting vehicles for control design. *ASME 2009 Dynamic Systems and Control Conference*. DSCC2009-2626: 181-188, https://doi.org/10.1115/DSCC2009-2626

Hindiyeh, R. Y., & Gerdes, J. C. (2014). A controller framework for autonomous drifting: Design, stability, and experimental validation. *Journal of Dynamic Systems, Measurement, and Control*, 136(5), DS-12-1023: 051015. https://doi.org/10.1115/1.4027471

Jazar, R. N. (2019). *Advanced vehicle dynamics*, Springer, New York

Milani, S., Marzbani, H., & Jazar, R. N. (2019). *Vehicle Drifting: Mathematical Theory and Dynamic Analysis*, *5th Recent Advances in Automotive Engineering* (ReCAR 2019), Mechanical Engineering Multi-Conference 2019, Universiti Kebangsaan Malaysia, Selangor, 20-22 August 2019

Shi, S., Li, L., Wang, X., Liu, H., Wang, Y. (2017). Analysis of the vehicle driving stability region based on the bifurcation of the driving torque and the steering angle. *Proc. Inst. Mech. Eng. Part D J. Automob. Eng.* 231, 984–998

Tavernini, D., Massaro, M., Velenis, E., Katzourakis, D. I., & Lot, R. (2013). Minimum time cornering: the effect of road surface and car transmission layout. *Vehicle System Dynamics*, 51(10), 1533-1547

Velenis, E., Frazzoli, E., & Tsiotras, P. (2010). Steady-state cornering equilibria and stabilisation for a vehicle during extreme operating conditions. *International Journal of Vehicle Autonomous Systems*, 8 (2-4), 217-241

Velenis, E., Katzourakis, D., Frazzoli, E., Tsiotras, P., & Happee, R. (2011). Steady-state drifting stabilization of RWD vehicles. *Control Engineering Practice*, 19(11), 1363-1376

Voser, C., Hindiyeh, R. Y., & Gerdes, J. C. (2010). Analysis and control of high sideslip manoeuvres. *Vehicle System Dynamics*, 48(S1), 317-336